MAT 335
Practice Problems

1. In old growth forests of Douglas fir, the spotted owl feasts mainly upon flying squirrels. If $\mathbf{p}_k = \begin{bmatrix} \mathbf{o}_k \\ \mathbf{s}_k \end{bmatrix}$ is a vector describing the population of owls and squirrels (in thousands) at the kth month and $\mathbf{p}_k = A\mathbf{p}_{k-1}$, with

$$A = \begin{bmatrix} .4 & .3 \\ -.4 & 1.2 \end{bmatrix}$$

a. Write a general form for $\mathbf{p}_k$ in terms of eigenvalues and eigenvectors of A.

$\mathbf{p}_k = c_1(1)^k\mathbf{x}_1 + c_2(0.6)^k\mathbf{x}_2$, where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 1 and $\mathbf{x}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 0.6

b. Describe what happens to the populations in the long run.

From above, $\mathbf{p}_k \to c_1\mathbf{x}_1$ as $k \to \infty$, so the population approaches a steady state with 1 owl for every 2000 squirrels.

2. Suppose $\mathbf{c} = \begin{bmatrix} \text{station A} \\ \text{station B} \\ \text{station C} \end{bmatrix}$ and $T = \begin{bmatrix} .83 & .15 & .35 \\ .07 & .73 & .25 \\ .10 & .12 & .40 \end{bmatrix}$ describes weekly transitions of rental cars from one station to another:

a. Diagonalize the matrix T (Find P and D such that $T = PDP^{-1}$).

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6697 & 0 \\ 0 & 0 & 0.2903 \end{bmatrix},\ P = \begin{bmatrix} 0.8673 & -0.7250 & -0.4190 \\ 0.4402 & 0.6878 & -0.3975 \\ 0.2326 & 0.0372 & 0.8164 \end{bmatrix}$$

b. Describe what happens to the voting distribution in the long run assuming T continues to describe transitions between groups.

Since $\mathbf{c}_k = T^k\mathbf{c}_0$ for any starting point $\mathbf{c}_0$, and $\mathbf{c}_k$ stabilizes on the first eigenvector, re-normalizing to sum to 1, we have the proportions approaching $\begin{bmatrix} .563 \\ .286 \\ .151 \end{bmatrix}$ as $k \to \infty$

c. For $P = [\mathbf{p_1}\ \mathbf{p_2}\ \mathbf{p_3}]$, then $\mathbf{c}_0 = a_1\mathbf{p_1} + a_2\mathbf{p_2} + a_3\mathbf{p_3}$ for some $a_1,\ a_2,\ a_3$, or $\mathbf{c}_0 = P\mathbf{a}$

and $\mathbf{a} = P^{-1}\mathbf{c}_0 = \begin{bmatrix} -519.4687 \\ 13.9951 \\ 96.3498 \end{bmatrix}$

so $\mathbf{c}_\infty = -519.4687 * \mathbf{p_1} = \begin{bmatrix} 450.5 \\ 228.7 \\ 120.8 \end{bmatrix}$

3. a. Based on the information given, if we define the population vector as $\mathbf{p}_k = \begin{bmatrix} c_k \\ y_k \\ a_k \end{bmatrix}$,

$$T = \begin{bmatrix} 0 & 0 & .42 \\ .60 & 0 & 0 \\ 0 & .75 & .95 \end{bmatrix}$$

b. Write a general expression for $\mathbf{p}_k$ in terms of eigenvalues and eigenvectors of T

$$\mathbf{p}_k = c_1(1.1048)^k\mathbf{x}_1 + c_2(\text{-}0.0774 - 0.4063i)^k\mathbf{x}_2 + c_3(\text{-}0.0774 + 0.4063i)^k\mathbf{x}_3,$$

where $\mathbf{x}_1 = \begin{bmatrix} 0.3489 \\ 0.1895 \\ 0.9178 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -0.0927 - 0.4867i \\ 0.7187 \\ -0.4537 + 0.1794i \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} -0.0927 + 0.4867i \\ 0.7187 \\ -0.4537 - 0.1794i \end{bmatrix}$

c. Show that the herd is growing, describe the growth rate and the relative proportion of calves, yearlings and adults in the female population in the long run

Given that the largest eigenvalue is greater than one, $\mathbf{p}_k$ grows without bound as $k \to \infty$, and $(1.1048)^k$ corresponds to growth rate of 10.48% per year (which will be the approximate rate for large k).

The proportions within the population will correspond to the first eignevector. Re-normalizing to a sum of 1 we have $\begin{bmatrix} 0.2396 \\ 0.1301 \\ 0.6303 \end{bmatrix}$